performance" satisfaction, is characterized as a truncated uniform distribution. This result was applied to the performance analysis of engagement duration and of several error sources. Linear theory enables the formulation of analytical solutions to the problem. When numerical Monte Carlo simulations are used to search for the worst-case distribution, a limited number of simulation runs typically are needed for the search. Moreover, when the miss distance is a concave function of the conflict duration on a certain interval, the worst-case miss distance can be determined from a singe Monte Carlo trial.

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Equations for Approximate Solutions Using Variational Calculus

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Introduction

TO develop differential calculus, neighboring points are used to derive the formulas for the derivatives of particular functions. Once these formulas are known, differentiation of complicated functions can be performed directly. There are many problems involving neighboring curves, for example, applications of regular perturbation theory, where Taylor series expansions are used to derive the formulas for variations of particular functionals. However, the step to taking variations has not been made. The purpose of this Note is to call attention to variational calculus.[1] Although the concepts of this Note apply to all types of equations (algebraic, differential, integral, etc.) and all types of problems (initial value, boundary value, etc.), the discussion is limited to initial-value problems involving nonlinear ordinary differential equations.

There are many avenues to the creation of an approximate analytical solution. Consider a single differential equation. 1) Approximations can be introduced into the differential equation and small terms simply discarded. If the resulting solution is accurate enough, work stops. If not, something more is needed. 2) Another possibility is to use error compensation,[2,3] that is, to replace the small terms in method 1 by a small constant to compensate for the errors. Then, by the using of the methods of regular perturbation theory,[4] a correction term can be found for the so-called zeroth-order solution in method 1. By the adjusting of the size of the small constant, the magnitude of the correction can be changed so that the approximate solution better fits the exact solution. 3) Another method is to find a small parameter in the differential equation and to use regular perturbation theory. This method has been used to obtain an approximate optimal control for the aeroassisted orbital plane change problem.[5] 4) Still another method is to assume that the solution lies in the neighborhood of a nominal solution. This is the approach that is used to obtain the Clohessy–Wiltshire equations for studying the motion of a mass relative to an orbiting mass.[6] Actually, methods 3 and 4 are similar in form; in method 3, the perturbed solution is caused by a perturbed parameter, whereas in method 4, the perturbed solution is caused by a perturbed initial condition.

Regardless of the method used, the standard procedure is to use Taylor series expansions to obtain the equations to be solved for the approximate solution. However, the expansion approach can be very involved, and often it is difficult to ensure that the resulting equations are correct. The messages of this Note are that the expansion approach and the variational approach[1] are equivalent, that the variational approach is simpler, and that it is helpful to have more than one way to derive the equations for the approximate solution to be sure they are correct.

In what follows, the expansion approach is reviewed briefly, the variational approach is discussed, and a satellite problem involving a small parameter is discussed to compare the two approaches. Finally, the Clohessy–Wiltshire equations are derived using variations.

Differential Equation and Initial Conditions

A problem in the realm of regular perturbation theory is to obtain an approximate analytical solution of a nonlinear ordinary differential equation having the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \varepsilon \boldsymbol{g}(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x}$ is an $n \times 1$ vector, the dot denotes a derivative with respect to time, and the scalar ε denotes a small parameter. The solution of the differential equation must satisfy the initial conditions

$$t_i = t_{i_s}, \qquad \boldsymbol{x}_i = \boldsymbol{x}_{i_s} \tag{2}$$

where the subscript s denotes a specific value.

It is assumed that a solution of the differential equation subject to the initial conditions exists. The functional form of the solution is given by

$$\boldsymbol{x} = \boldsymbol{x}(t, \varepsilon, t_i, \boldsymbol{x}_i) \tag{3}$$

For the case where the initial conditions are not to be changed, the solution can be rewritten in the form

$$\boldsymbol{x} = \boldsymbol{x}(t, \varepsilon) \tag{4}$$

Taylor Series Approach

Expanding Eq. (4) in terms of the small parameter ε leads to

$$\boldsymbol{x} = \boldsymbol{x}(t, 0) + \boldsymbol{x}_\varepsilon(t, 0)\varepsilon + \tfrac{1}{2!}\boldsymbol{x}_{\varepsilon\varepsilon}(t, 0)\varepsilon^2 + \cdots \tag{5}$$

where the subscript ε denotes a partial derivative with respect to ε. Hence, the solution of Eq. (1) can be assumed to have the form

$$\boldsymbol{x} = \boldsymbol{x}_0(t) + \boldsymbol{x}_1(t)\varepsilon + \boldsymbol{x}_2(t)\varepsilon^2 + \cdots \tag{6}$$

In practice, the Taylor series approach is to substitute Eq. (6) into a particular differential equation and initial conditions, carry out whatever mathematical operations and expansions are needed to write the equations in terms of powers of ε, and equate the coefficient of each power of ε to zero to obtain the differential equations and initial conditions for $x_0, x_1, x_2, \ldots$.

To illustrate the Taylor series approach, consider a satellite moving in the equatorial plane of an oblate spheroid Earth. The motion of the satellite is governed by the equation[7]

$$\ddot{r} - h^2/r^3 = -(\mu/r^2)[1 + (R^2/r^2)\varepsilon] \tag{7}$$

where r is the radial distance to the satellite, h is the constant angular momentum of the satellite, R is the radius of the Earth, μ is the gravitational constant, and $\varepsilon = (3/2)J_2 = 0.001624$ is the flatness. The initial conditions are given by

$$t_i = t_{i_s}, \qquad r_i = r_{i_s}, \qquad \dot{r}_i = \dot{r}_{i_s} \tag{8}$$

Received 9 September 1999; revision received 3 April 2000; accepted for publication 4 May 2000.

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The objective is to derive the equations that govern the approximate solution of Eq. (7) using ε as a small parameter. This problem could be discussed using two first-order equations, but it is easier to leave it as a second-order equation.

Assume that

$$r = r_0(t) + r_1(t)\varepsilon + r_2(t)\varepsilon^2 + \cdots = r_0 + \Delta r = r_0(1 + \Delta r/r_0) \tag{9}$$

so that Eq. (7) becomes

$$\ddot{r}_0 + \frac{d^2\Delta r}{dt^2} - \frac{h^2}{r_0^3}\left(1 + \frac{\Delta r}{r_0}\right)^{-3} = -\frac{\mu}{r_0^2}\left(1 + \frac{\Delta r}{r_0}\right)^{-2}\left[1 + \left(\frac{R}{r_0}\right)^2\left(1 + \frac{\Delta r}{r_0}\right)^{-2}\varepsilon\right] \tag{10}$$

Then, with the binomial expansion

$$(1 + x)^n = 1 + nx + [n(n-1)/2!]x^2 + \cdots \tag{11}$$

Equation (10) can be rewritten as

$$\ddot{r}_0 + \frac{d^2\Delta r}{dt^2} - \frac{h^2}{r_0^3} + \frac{3h^2\Delta r}{r_0^4} + \cdots = -\frac{\mu}{r_0^2}\left[1 - \frac{2\Delta r}{r_0} + \left(\frac{R}{r_0}\right)^2\varepsilon + \cdots\right] \tag{12}$$

Finally, with $\Delta r = r_1\varepsilon + r_2\varepsilon^2 + \cdots$ and setting the coefficients of the various powers of ε to zero, the various-order solutions are governed by the equations

$$\begin{aligned} \ddot{r}_0 - h^2/r_0^3 &= -\left(\mu/r_0^2\right) \\ \ddot{r}_1 + \left[(3h^2 - 2\mu r_0)/r_0^4\right]r_1 &= -\left(\mu R^2/r_0^4\right) \\ &\vdots \end{aligned} \tag{13}$$

Substitution of Eq. (9) into Eq. (8) leads to the initial conditions for $r_0, r_1, \ldots,$

$$\begin{aligned} t_i &= t_{i_s} \\ (r_0)_i &= r_{i_s}, \qquad (\dot{r}_0)_i = \dot{r}_{i_s} \\ (r_1)_i &= 0, \qquad (\dot{r}_1)_i = 0 \\ &\vdots \end{aligned} \tag{14}$$

Variational Approach

In differential calculus, a differential is formed by evaluating a function such as $x(t)$ at neighboring points on the same curve, that is, $x(t_*) - x(t)$, and letting the two points approach each other. In variational calculus, a variation is formed by evaluating functions at corresponding points (at the same value of the time) on neighboring curves, for example, $x_*(t) - x(t)$. Whereas the differential is an infinitesimal, the variation is a displacement.

Relative to the differential equation (1), let $\boldsymbol{x}$ denote the nominal solution $\boldsymbol{x}(t, 0)$ and $\boldsymbol{x}_*$ denote the neighboring or perturbed solution $\boldsymbol{x}(t, \varepsilon)$. The solution $\boldsymbol{x}_*$ has a zeroth-order part $\boldsymbol{x}$, a first-order part $\tilde{\delta}\boldsymbol{x}$, a second-order part $\tilde{\delta}^2\boldsymbol{x}, \ldots$, and is written as

$$\boldsymbol{x}_* = \boldsymbol{x} + \tilde{\delta}\boldsymbol{x} + \tfrac{1}{2!}\tilde{\delta}^2\boldsymbol{x} + \cdots \tag{15}$$

where the tilde emphasizes that $\boldsymbol{x}_*$ and $\boldsymbol{x}$ are being evaluated at the same value of the time. The time-fixed variation $\tilde{\delta}$ behaves like a differential,[1] implying that

$$\tilde{\delta}(\boldsymbol{x}) = \tilde{\delta}\boldsymbol{x}, \qquad \tilde{\delta}(\tilde{\delta}\boldsymbol{x}) = \tilde{\delta}^2\boldsymbol{x}, \ldots \tag{16}$$

and the processes of variation and differentiation interchange,[1] that is,

$$\tilde{\delta}(d\boldsymbol{x}) = d(\tilde{\delta}\boldsymbol{x}) \tag{17}$$

The relationship between the two solutions $\boldsymbol{x}_*$ and $\boldsymbol{x}$ is given by $\tilde{\delta}\boldsymbol{x}, \tilde{\delta}^2\boldsymbol{x}, \ldots$. In practice, the differential equations and initial conditions for $\tilde{\delta}\boldsymbol{x}, \tilde{\delta}^2\boldsymbol{x}, \ldots$ are obtained by taking the variations of the particular differential equation and its initial conditions.

To illustrate the procedure, consider the differential equation (7) and the initial conditions (8). The variational approach begins with taking the time-fixed variation of Eq. (7), that is,

$$\tilde{\delta}\ddot{r} + (3h^2/r^4)\tilde{\delta}r = (2\mu/r^3)\tilde{\delta}r + (4\mu R^2/r^5)\varepsilon\tilde{\delta}r - (\mu R^2/r^4)\tilde{\delta}\varepsilon \tag{18}$$

Because ε is a constant, the tilde on $\tilde{\delta}\varepsilon$ can be dropped. With $\varepsilon = 0$, Eq. (7) gives the equation for the zeroth-order solution to be

$$\ddot{r} - h^2/r^3 = -(\mu/r^3) \tag{19}$$

and Eq. (18) gives the equation for the first-order solution, that is,

$$\frac{d^2\tilde{\delta}r}{dt^2} + \frac{3h^2 - 2\mu r}{r^4}\tilde{\delta}r = -\frac{\mu}{r^4}R^2\delta\varepsilon \tag{20}$$

after $\tilde{\delta}$ and d are interchanged. The equation for $\tilde{\delta}^2 r$ can be derived by taking the variation of Eq. (18) with $\tilde{\delta}(\delta\varepsilon) = 0$ and setting $\varepsilon = 0$. When the solutions of all orders have been obtained, the overall solution r_* becomes

$$r_* = r + \tilde{\delta}r + \tfrac{1}{2!}\tilde{\delta}^2 r + \cdots \tag{21}$$

The equivalence of the expansion approach and the variational approach is established by making the following connections:

$$\varepsilon \to \delta\varepsilon, \qquad r_0 \to r, \qquad r_1\varepsilon \to \tilde{\delta}r, \qquad r_2\varepsilon^2 \to \tfrac{1}{2!}\tilde{\delta}^2 r, \ldots \tag{22}$$

The initial conditions for these equations are obtained by taking the variation of Eq. (8). They are given by

$$\begin{aligned} t_i &= t_{i_s} \\ r_i &= r_{i_s}, \qquad \dot{r}_i = \dot{r}_{i_s} \\ (\tilde{\delta}r)_i &= 0, \qquad (\tilde{\delta}\dot{r})_i = 0 \\ &\vdots \end{aligned} \tag{23}$$

Note that the Taylor series approach requires the binomial expansion and the multiplication of several series, whereas the variational approach just requires differentiation. Also, with the variational approach, the creation of the equations for another order is easily accomplished. It is not so with the Taylor series approach.

Clohessy–Wiltshire Equations

The Clohessy–Wiltshire (CW) equations[5] are the small perturbation equations for motion relative to a point moving on a prescribed orbit. They are derived here for a circular orbit using the variational approach. For planar motion, the nonlinear equations are given by

$$\begin{aligned} \ddot{x} - 2\omega\dot{z} - \omega^2 x + \mu(x/r^3) &= 0 \\ \ddot{z} + 2\omega\dot{x} - \omega^2(z - r_0) + \mu\left[(z - r_0)/r^3\right] &= 0 \end{aligned} \tag{24}$$

where

$$r = \sqrt{x^2 + (z - r_0)^2} \tag{25}$$

In these formulas, μ is the gravitational constant, x and z are the coordinates of a point mass in the local-horizontal local-vertical reference frame whose origin is moving on a circular orbit, and r is the distance from the point mass to the center of the Earth. The radius of the orbit is r_0, and

$$\omega^2 = \mu/r_0^3 \tag{26}$$

The initial conditions of the point mass are given by

$$x_i = \xi, \qquad z_i = 0, \qquad \dot{x}_i = 0, \qquad \dot{z}_i = 0 \tag{27}$$

where ξ is a small perturbation in x_i. The objective is to derive the small perturbation equations of motion for the point mass using variations.

Taking the time-fixed variation of Eqs. (22) leads to

$$\tilde{\delta}\ddot{x} - 2\omega\tilde{\delta}\dot{z} - \omega^2\tilde{\delta}x + \mu[(r^3\tilde{\delta}x - x3r^2\tilde{\delta}r)/r^6] = 0$$

$$\tilde{\delta}\ddot{z} + 2\omega\tilde{\delta}\dot{x} - \omega^2\tilde{\delta}z + \mu\left\{\left[r^3\tilde{\delta}z - (z - r_0)3r^2\tilde{\delta}r\right]\Big/r^6\right\} = 0 \tag{28}$$

where

$$\tilde{\delta}r = [x\tilde{\delta}x + (z - r_0)\tilde{\delta}z]/r \tag{29}$$

Next, the coefficients of the variations are evaluated on the reference orbit ($x = z = 0$ and $r = r_0$) so that $\tilde{\delta}r = -\tilde{\delta}z$, and the CW equations become

$$\frac{d^2}{dt^2}\tilde{\delta}x - 2\omega\frac{d}{dt}\tilde{\delta}z = 0, \qquad \frac{d^2}{dt^2}\tilde{\delta}z + 2\omega\frac{d}{dt}\tilde{\delta}x - 3\omega^2\tilde{\delta}z = 0 \tag{30}$$

Finally, the initial conditions are obtained by taking the time-fixed variation of Eq. (27) and are given by

$$(\tilde{\delta}x)_i = \delta\xi, \qquad (\tilde{\delta}z)_i = 0, \qquad (\tilde{\delta}\dot{x})_i = 0, \qquad (\tilde{\delta}\dot{z})_i = 0 \tag{31}$$

If second-order terms are needed, they can be obtained by taking the variation of Eqs. (28) and (31) with $\tilde{\delta}(\delta\xi) = 0$. Once the various-order solutions have been obtained, the overall solution is given by

$$x_* = x + \tilde{\delta}x + \tfrac{1}{2!}\tilde{\delta}^2x + \cdots, \qquad z_* = z + \tilde{\delta}z + \tfrac{1}{2!}\tilde{\delta}^2z + \cdots \tag{32}$$

where x and z are zero.

Conclusions

The equations defining the various-order solutions of a problem in the realm of regular perturbation theory have been derived by using Taylor series expansions and by using variational calculus. It has been shown that the two approaches are equivalent and that the variational approach is easier to use. Variational calculus only requires differentiation, and the equations for higher-order solutions can be obtained from earlier-order equations. The same comments apply to perturbation problems involving perturbed initial conditions. Although variational calculus has only been discussed for initial-value problems, it also applies to boundary-value problems, but both time-fixed and time-free variations as well as the relationship between them are required. In fact, variational calculus applies to all problems involving neighboring curves or surfaces, for that matter.

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Improved Dispersion of a Fin-Stabilized Projectile Using a Passive Movable Nose

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Introduction

THE merit of a penetrator is often assessed by a relatively short list of metrics that typically includes parameters such as terminal velocity, penetrator weight, cost, system accuracy, etc. Of these parameters, system accuracy is usually near the top of the list in terms of importance and of significant concern during weapon system development. Given two identical weapon systems with the exception of accuracy, the system with superior accuracy enjoys a distinct advantage on the battlefield. A system with improved accuracy can engage targets at a greater range and obtain the same probability of hit, providing the tank commander with increased flexibility during an engagement. Alternatively, a system with better accuracy will register more first volley hits at the same range, reducing the counterfire threat. Furthermore, a gun system with superior accuracy ultimately requires fewer shots to achieve mission objectives, hence inducing less burden on the logistics pipeline.

The initial state of a projectile as it exits the gun muzzle and enters free flight can be viewed as a random process. The random nature of the initial free-flight state stems from many effects but perhaps most notably from gun tube and projectile manufacturing tolerances combined with resulting gun tube and projectile vibration. As the projectile flies downrange, these uncertainties, along with aerodynamic disturbances along the trajectory, map into dispersion at the target. A designer can take two basic approaches toward improving accuracy, that is, reduce the variability of projectile initial free-flight conditions or reduce the sensitivity of the projectile trajectory to initial free-flight conditions. One way to attack this problem using the latter approach is to replace the rigid wind screen with a passive gimballed nose. If the pivot point of the nose section is forward of the nose aerodynamic center, then the nose will tend to rotate into the relative wind and subsequently reduce aerodynamic jump caused by projectile normal force. A passive gimballed nose projectile is an attractive design modification because it is a relatively simple mechanism that requires no active electronic controls. Furthermore, for many penetrator designs, the nose cone is empty and could easily house the gimbal joint.

Early in the development of controlled rockets, the notion of utilizing a movable nose to control the trajectory of a projectile actively was established.[1] Goddard obtained a patent that outlined the basic concept. More recently, Barrett and Stutts[2] further developed this concept. The movable nose concept has also been investigated in unguided projectile applications as well. Kranz[3] obtained a patent for a telescopic gimballed nose on a high-velocity aerodynamic body. Schmidt and Donovan[4] developed a simple closed-form solution for an effective $C_{L\alpha}$ and $C_{M\alpha}$ for a movable nose projectile configuration that is based on projectile linear theory.[5] A limited number of prototype projectiles were fired, and range data were reduced to estimate aerodynamic coefficients. The work reported here extends the previous work mentioned by simulating the exterior ballistics of a gimballed nose projectile in atmospheric flight and subsequently comparing impact point dispersion statistics with a similarly sized rigid projectile.

Received 10 April 2000; revision received 10 May 2000; accepted for publication 10 May 2000.

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